

Smarr formula for BTZ black holes in general three-dimensional gravity models

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Abstract

Recent studies have presented the interpretation of thermodynamic enthalpy for the mass of BTZ black holes and the corresponding Smarr formula. All these are made in the background of three-dimensional (3D) general relativity. In this paper, we extend such interpretation into general 3D gravity models. It is found that the direct extension is unfeasible and some extra conditions are required to preserve both the Smarr formula and the first law of black hole thermodynamics. Thus, BTZ black hole thermodynamics enforces some constraints for general 3D gravity models, and these constraints are consistent with all previous discussions.

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I. INTRODUCTION

Smarr formula [1], together with the first law of black hole thermodynamics [2], plays an important role in black hole physics. For black hole solutions with a non-vanishing cosmological constant, in order to maintain the scaling relation of Smarr formula, the cosmological constant is considered as thermodynamic pressure [3–9], and the black hole mass, which is usually regarded as analogous thermal internal energy in black hole thermodynamics, is interpreted as a gravitational analog of thermodynamic enthalpy H [10], i.e. $M = H = U + PV$, where U is the internal energy and the pressure P is derived from the cosmological constant. This has been investigated for many different situations (see Ref. [11] and references therein), and the consistent expressions for homogeneous Smarr formula and the first law have been constructed, i.e. for a D -dimensional singly-rotating black hole,

$$dM = TdS + \Omega dJ + VdP, \quad (1)$$

$$(D - 3) M = (D - 2) TS + (D - 2) \Omega J - 2VP, \quad (2)$$

where T is its temperature, S is its entropy, J is its angular momentum, Ω is its angular velocity, V is its thermodynamic volume, and $P = -\frac{\Lambda}{8\pi} = \frac{(D-2)(D-1)}{16\pi l^2}$ is its pressure.

Recently, such interpretation as presented in Eqs. (1) and (2) has also been investigated for lower-dimensional black holes under the background of extended thermodynamic phase space, or “black hole chemistry” [12]. For three-dimensional (3D) Banados-Teitelboim-Zanelli (BTZ) black holes [13] which is an solution of 3D Einstein field equation plus a negative cosmological constant, it has been confirmed [12, 14] that Eqs. (1) and (2) can be satisfied simultaneously by assuming the negative cosmological constant as a variable thermodynamic parameter. It was found that the extension to the charged BTZ black holes is direct [12], but to ensure the Reverse Isoperimetric Inequality [8, 15] a new thermodynamic work term associated with the mass-renormalization scale has to be introduced. Moreover, the mass of BTZ black holes as the enthalpy of the spacetime was also found to be valid even under the condition of quantum correction [5].

It is realized that the preservation of Eqs. (1) and (2) was only investigated under 3D general relativity (GR), however. 3D GR propagates no physical modes (it has no local degrees of freedom) and can be formulated as a Chern-Simons (CS) gauge theory [16]. The attempt to construct generalizations [17–22] of 3D GR that propagate gravitons has

been a revival of interest in past years. In these new 3D gravity models, some of the thermodynamic parameters for BTZ black holes which almost solves all the 3D gravity models have to be modified, and the modified parameters still satisfied the first law of black hole thermodynamics and the integral mass formula [23] but without forcing the cosmological constant into a variable. Of course, the previous Smarr-like formula, i.e. $M = \frac{1}{2}TS + \Omega J$, does not satisfy the scaling relation as presented in Eq. (2). In this paper, we expect to investigate the relations among the thermodynamic parameters under the background of extended 3D gravity models, and see if there are the same relation as that in Eqs. (1) and (2) when considering the cosmological constant as a new variable thermodynamic parameter. Moreover, a motivation to make such investigation is that there are the phenomena of continuous phase transition [24–26] for BTZ black holes in general 3D gravity models.

The organization of this paper is as follows. First, we revisit the interpretation of thermodynamic enthalpy for the mass of BTZ black holes and the homogeneous Smarr formula in the background of 3D normal Einstein gravity in the second section. In the third section, we extend these discussions to the case of 3D exotic Einstein gravity. Then, we investigate these relations systematically in general 3D gravity models and see if they are still valid, and investigate the phase transition for this case in the fourth section. Finally, we summarize our conclusion in the fifth section.

II. NORMAL BTZ BLACK HOLES

The two-parameter BTZ solution represents a rotating black hole in 3D spacetime with a negative cosmological constant, and its metric takes the form

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \quad (3)$$

where ϕ is an angle with the period 2π as the identification of the black hole spacetime. The functions N^2 and N^ϕ are

$$N^2 = -8Gm + \frac{r^2}{\ell^2} + \frac{16G^2 j^2}{r^2}, N^\phi = \frac{4Gj}{r^2}, \quad (4)$$

where G the 3D Newton constant. Its Killing horizons are found by setting $N^2 = 0$; this gives

$$r_\pm = \sqrt{2G\ell(\ell m + j)} \pm \sqrt{2G\ell(\ell m - j)}, \quad (5)$$

We may assume without loss of generality that $j \geq 0$ and assume that $\ell m \geq j$, to ensure the existence of an event horizon at $r = r_+$. Thus, the parameters m and j can be expressed as

$$m = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad j = \frac{r_+ r_-}{4G\ell}. \quad (6)$$

And the temperature T and the angular velocity Ω take the values

$$T = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell^2}, \quad \Omega = \frac{r_-}{\ell r_+} \quad (7)$$

which are independent on the concrete 3D gravity models and present the geometric properties for the 3D BTZ spacetime background.

The BTZ metric can be solved in the normal Einstein gravity with the Lagrangian

$$L_G = \frac{1}{8\pi G} \left(e_a R^a + \frac{\Lambda_0}{6} \epsilon^{abc} e_a e_b e_c \right), \quad (8)$$

where the Lagrangians are given with 3-forms in which e^a ($a = 0, 1, 2$) is the dreibein 1-forms, ω^a is Lorentz connection 1-forms, $\Lambda_0 = -\frac{1}{\ell^2}$ is the negative cosmological constant and $R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c$ is the curvature 2-form field strengths. With the BTZ metric (3), the thermodynamic parameters for normal BTZ black holes are

$$M_G = m, \quad J_G = j, \quad S_G = \frac{\pi r_+}{2G}. \quad (9)$$

It is easy to confirm that the usual thermodynamic relations, $dM_G = TdS_G + \Omega dJ_G$ and $M_G = \frac{1}{2}TS_G + \Omega J_G$, hold in this situation, but the mass formula obviously violates the scaling relation required in the Smarr formula of rotating black holes. In order to rectify this situation, it is possible to promote the cosmological constant to a variable thermodynamic parameter, so that the corresponding relations from Eqs. (1) and (2) can be reexpressed as

$$\begin{aligned} dM_G &= TdS_G + \Omega dJ_G + V_G dP, \\ 0 &= TS_G + \Omega J_G - 2V_G P, \end{aligned} \quad (10)$$

where the pressure is $P \equiv \frac{1}{8\pi G\ell^2}$ and the conjugate thermodynamic volume is $V_G = \frac{\partial M_G}{\partial P} = \pi r_+^2$ obtained by the mass $M(S_G, J_G, P) = \frac{4PS_G^2}{\pi} - \frac{\pi^2 J_G^2}{128S_G^2}$. It is noted that the thermodynamic volume is just the 3D geometric volume, independent on other thermodynamic parameters except the entropy.

Like AdS black holes in higher dimensions [10], the mass of BTZ black holes should be understood as thermodynamic enthalpy when the negative cosmological constant is considered as a thermodynamic variable, but it is noted that the thermodynamic enthalpy is zero

here. This can be understood by the fact [30] that the energy required to form the BTZ black hole is balanced by the energy required by the external forces to place the black hole into the cosmological environment.

III. EXOTIC BTZ BLACK HOLES

The BTZ metric can also be solved in the exotic Einstein gravity [27] with the Lagrangian

$$L_E = \frac{\ell}{8\pi G} \left[\omega_a \left(d\omega^a + \frac{2}{3} \epsilon^{abc} \omega_b \omega_c \right) - \frac{1}{\ell^2} e_a T^a \right], \quad (11)$$

where the torsion tensor is $T^a = de^a + \epsilon^{abc} \omega_b e_c$, but the corresponding thermodynamic parameters becomes

$$M_E = j/\ell, \quad J_E = \ell m, \quad S_E = \frac{\pi r_-}{2G}. \quad (12)$$

This presents some exotic properties: the metric is the same as that of 3D Einstein gravity but with reversed roles for mass and angular momentum, and the entropy is proportional to the length of the inner horizon instead of the event horizon. The corresponding understanding for these properties refer to Refs. [28, 29]. It is easy to check that the usual thermodynamic relations, $dM_E = TdS_E + \Omega dJ_E$ and $M_E = \frac{1}{2}TS_E + \Omega J_E$, also hold in this situation.

As done for normal BTZ black holes, we also promote the cosmological constant into a variable in this case to see if the Smarr formula can also obtained like before. Based on this, now we express the mass of exotic BTZ black holes as

$$M_E = \frac{1}{\ell} \left(\frac{2GS_E^2 J_E}{\pi^2 \ell} - \frac{G^2 S_E^4}{\pi^4 \ell^2} \right)^{\frac{1}{2}}. \quad (13)$$

When considering the pressure $P \equiv \frac{1}{8\pi G \ell^2}$, the conjugate thermodynamic volume is obtained as

$$\begin{aligned} V_E &= \frac{\partial M_E}{\partial P} = \frac{\partial M_E / \partial \ell}{\partial P / \partial \ell} \\ &= -\frac{3}{2\ell^2} \left(\frac{2GS_E^2 J_E}{\pi^2 \ell} - \frac{G^2 S_E^4}{\pi^4 \ell^2} \right)^{\frac{1}{2}} + \frac{1}{2\ell^2} \left(\frac{2GS_E^2 J_E}{\pi^2 \ell} - \frac{G^2 S_E^4}{\pi^4 \ell^2} \right)^{-\frac{1}{2}} \\ &= \frac{\pi}{2} \frac{3r_+^2 r_- - r_-^3}{r_+} \end{aligned} \quad (14)$$

where the last step uses the relation in Eqs. (6) and (12). It is noted that the exotic volume is not the geometric volume $V = \pi r_+^2$ and it also depends on the thermodynamic parameters J_E and P besides the entropy S_E .

With the expressions of thermodynamic pressure and volume, we obtain the relations again

$$\begin{aligned} dM_E &= TdS_E + \Omega dJ_E + V_E dP, \\ 0 &= TS_E + \Omega J_E - 2V_E P, \end{aligned} \tag{15}$$

which presents the homogenous Smarr relation.

On the other hand, one might want to make the thermodynamic volume unchanged, i.e. $V_E = V = \pi r_+^2$, whose thoughts stemmed from the geometric expressions for the temperature and the angular velocity. Thus the Smarr formula is written as

$$0 = TS_E + \Omega J_E - 2V P_E, \tag{16}$$

where a calculation leads to the pressure $P_E = \frac{3r_+^2 r_- - r_-^3}{16\pi G \ell^2 r_+^2}$. But unfortunately, this cannot preserve the first law, i.e. $dM_E \neq TdS_E + \Omega dJ_E + V dP_E$, which can be obtained only through the calculation of the derivative to the AdS radius ℓ . On the other hand, such expression for the pressure implies that P_E is not an independent variable relative to the entropy and the angular momentum. This conflicts with the requirement of differential form of the first law of black hole thermodynamics.

IV. BTZ BLACK HOLES IN GENERAL 3D GRAVITY MODELS

As seen above for the normal and exotic BTZ black holes that the extensive thermodynamic variables are model-dependent, so in general 3D gravity models, the mass and the angular momentum are expressed as

$$\begin{aligned} M &= am + bj/\ell, \\ J &= aj + b\ell m, \\ S &= \frac{\pi}{2G} (ar_+ + br_-), \end{aligned} \tag{17}$$

where a and b are the parameters dependent on the concrete models, i.e. for normal Einstein gravity, $a = 1, b = 0$; for exotic Einstein gravity, $a = 0, b = 1$. If the parameters a and b

are independent on the cosmological constant or the AdS radius ℓ , we can get the same expressions

$$\begin{aligned} dM &= TdS + \Omega dJ + VdP, \\ 0 &= TS + \Omega J + 2VP, \end{aligned} \tag{18}$$

with the thermodynamic volume

$$V = \frac{2ar_+^3 + 3br_+^2r_- - br_-^3}{2r_+}\pi \tag{19}$$

which is not geometric, but will recover geometric volume when $a = 1, b = 0$. It is also obvious that when $a = 0, b = 1$, $V = V_E$.

Then, in general 3D gravity models, are the parameters a and b independent on the AdS radius? As well-known, BTZ black holes are the solutions almost for all the 3D gravity models which are usually modified by topological terms, i.e. topological massive gravity (TMG) [17], minimal massive gravity (MMG) [22], which make the theory has one gravitational propagating mode, or by higher-order derivative terms, i.e. new massive gravity (NMG) [19], zwei-dreibein gravity (ZDG) [21], which make the theory have two gravitational propagating modes, or by the two terms simultaneously, i.e. general massive gravity (GMG) [20], general minimal massive gravity (GMMG) [31], or by other terms, i.e. Mielke–Baekler (MB) model [18], et al. But for almost all these modified 3D gravity models, either a or b or both of them are dependent on the AdS radius. Thus, we have to check if the homogenous Smarr formula is still valid when taking $a = a(\ell), b = b(\ell)$.

Assuming the Smarr formula in Eq. (18) holds, we can obtain the same expression for thermodynamic volume as in Eq. (19) but the parameters a and b are now the function of AdS radius ℓ . For the first law of black hole thermodynamics, we have

$$\frac{dM}{d\ell} = \frac{r_+^2 + r_-^2}{8G\ell^2} \frac{da}{d\ell} + \frac{r_+r_-}{4G\ell^2} \frac{db}{d\ell} - \frac{r_+^2 + r_-^2}{4G\ell^3} a - \frac{r_+r_-}{2G\ell^3} b, \tag{20}$$

and

$$\begin{aligned} T \frac{dS}{d\ell} &= \frac{r_+^2 - r_-^2}{4G\ell^2} \frac{da}{d\ell} + \frac{r_- (r_+^2 - r_-^2)}{4G\ell^2 r_+} \frac{db}{d\ell}, \\ \Omega \frac{dJ}{d\ell} &= \frac{r_-^2}{4G\ell^2} \frac{da}{d\ell} + \frac{r_- (r_+^2 + r_-^2)}{8G\ell^2 r_+} \frac{db}{d\ell} - \frac{r_-^2}{4G\ell^3} a - \frac{r_- (r_+^2 + r_-^2)}{8G\ell^3 r_+} b, \\ V \frac{dP}{d\ell} &= -\frac{r_+^2}{4G\ell^3} a - \frac{r_- (3r_+^2 - r_-^2)}{8G\ell^3 r_+} b. \end{aligned}$$

Thus, it is easy to see that

$$\frac{dM}{d\ell} \neq T \frac{dS}{d\ell} + \Omega \frac{dJ}{d\ell} + V \frac{dP}{d\ell}, \quad (21)$$

which indicates that it is impossible to find the proper thermodynamic volume to maintain the first law and the homogenous Smarr formula simultaneously if the cosmological constant is taken as the pressure $P \equiv \frac{1}{8\pi G \ell^2}$. This seems that the interpretation of enthalpy was inapplicable for the 3D AdS black hole in the general gravity models.

When we deal with the concrete gravity model, however, it is found that the simultaneous preservation of the first law and the homogeneous Smarr formula will give some further constraints when the parameters a and b are dependent on the AdS radius. But whether the constraints are reasonable has to be checked. Here we take the GMMG [31] for example to study this and start with its Lagrangian

$$L_M = L_G + \frac{1}{2\mu} L_{CS} + \frac{1}{m^2} L_H + h_a T^a + \frac{\alpha}{2} \epsilon^{abc} e_a h_b h_c \quad (22)$$

where $L_{CS} = \frac{1}{8\pi G} \omega_a (d\omega^a + \frac{2}{3} \epsilon^{abc} \omega_b \omega_c)$ is the topological modification term, $L_H = \frac{1}{8\pi G} (f^a R_a + \frac{1}{2} \epsilon^{abc} e_a f_b f_c)$ is the higher-order derivative modification term, and μ, m, α are the parameters which are introduced in TMG, NMG and MMG. For GMMG model, it was obtained [32]

$$a = 1 + \frac{\gamma}{2\mu^2 \ell^2} + \frac{s}{2m^2 \ell^2}, \quad b = \frac{1}{\mu \ell}, \quad (23)$$

where γ, s are the constants. Through a tedious calculation, it is found that only if

$$\mu \ell = \text{constant and } m \ell = \text{constant}, \quad (24)$$

we can ensure both the first law and the homogeneous Smarr formula, which means that the interpretation of thermodynamic enthalpy for BTZ black holes in general gravity models enforces the extra constraints as in Eq. (24) when we promote the cosmological constant into a variable thermodynamic parameter.

Now we try to understand why there is such constraints as in Eq. (24). First, these constraints are consistent with the previous discussions in different 3D gravity models, in particular for some special situations, i.e. $\mu \ell = 1$ in TMG model leads to 3D chiral gravity [33]; $m^2 \ell^2 = \frac{1}{2}$ (or $\Lambda_0/m^2 = -1$) in NMG leads to an extra gauge symmetry at the linearized level which allows massive modes to become partially massless [34].

On the other hand, the BTZ metric is locally isomorphic to the AdS vacuum, so any theory of 3D gravity admitting an AdS vacuum will also admit BTZ black holes. Now

we search for the AdS vacuum for GMMG models which is also the maximally symmetric vacuum [35] of GMMG defined by

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}. \quad (25)$$

Thus, the GMMG field equation

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} + \frac{s}{2m^2} K_{\mu\nu} = 0, \quad (26)$$

where $G_{\mu\nu}$ is Einstein tensor, $C_{\mu\nu} = \frac{1}{\sqrt{-g}} \varepsilon_{\mu\alpha\beta} \nabla^\alpha (R_\nu^\beta - \frac{1}{4} \delta_\nu^\beta R)$ is Cotton tensor, and $J_{\mu\nu}$, $K_{\mu\nu}$ are the higher-order modification terms (see Ref. [33] for their expressions), will be reduced to

$$\Lambda_0 - \Lambda + \frac{\gamma \Lambda^2}{4\mu^2} - \frac{s \Lambda^2}{4m^2} = 0, \quad (27)$$

which solves the effective cosmological constant

$$\Lambda = \frac{2 \left(1 \pm \sqrt{1 - \Lambda_0 \left(\frac{\gamma}{\mu^2} - \frac{s}{m^2} \right)} \right)}{\frac{\gamma}{\mu^2} - \frac{s}{m^2}}. \quad (28)$$

It is consistent with that obtained in the linearization of GMMG field equation around the AdS metric [32]. The AdS space requires the effective cosmological constant $\Lambda = -\frac{1}{\ell^2} < 0$. This enforces the coupling parameters to satisfy certain conditions, i.e. the minus sign for the sign “ \pm ” can be taken but must $\frac{\gamma}{\mu^2} - \frac{s}{m^2} > 0$, $\Lambda_0 < 0$. Thus, we find the AdS vacuum for GMMG models with $\Lambda = -\frac{1}{\ell^2}$, which means that we must also find BTZ black holes because these are locally isometric to the AdS vacuum.

On the other hand, Ref. [10] has extended the Komar integral relation for the asymptotically flat black holes to that for the asymptotically AdS black holes,

$$\frac{1}{8\pi G} \int_{\partial\Sigma} dS_{ab} (\nabla^a \xi^b + 2\Lambda \omega^{ab}) = 0, \quad (29)$$

where dS_{ab} is the volume element normal to the co-dimension 2 surface $\partial\Sigma$ which is the boundary of the hypersurface Σ in BTZ black hole spacetime, ξ^a is the Killing vector on this spacetime, and ω^{ab} is the anti-symmetric Killing potential by solving $\xi^b = \nabla_a \omega^{ab}$. For the boundary $\partial\Sigma = \partial\Sigma_\infty \cap \partial\Sigma_{r_+}$, one can rearrange the Komar integral,

$$\frac{1}{8\pi G} \int_{\partial\Sigma_\infty} dS_{ab} (\nabla^a \xi^b + 2\Lambda \omega^{ab}) = \frac{1}{8\pi G} \int_{\partial\Sigma_{r_+}} dS_{ab} (\nabla^a \xi^b + 2\Lambda \omega^{ab}), \quad (30)$$

which leads to the homogeneous Smarr formula directly for normal BTZ black holes [12]. However, in general 3D gravity models, when we promote the cosmological constant Λ into a variable, those coupling constants, i.e. μ , m , will also become variables, as seen from Eq. (28). In order to ensure both the thermodynamic first law and the homogeneous Smarr formula to hold, one must constraint these coupling parameters to satisfy the relation in Eq. (24), which might be the justification for these constraints.

Moreover, from the left and right central charges of dual CFT [31] of GMMG model,

$$\begin{aligned} C_L &= \frac{3\ell}{2G} \left(1 - \frac{1}{\mu\ell} + \frac{\gamma}{2\mu^2\ell^2} + \frac{s}{2m^2\ell^2} \right), \\ C_R &= \frac{3\ell}{2G} \left(1 + \frac{1}{\mu\ell} + \frac{\gamma}{2\mu^2\ell^2} + \frac{s}{2m^2\ell^2} \right), \end{aligned} \quad (31)$$

the microscopic Cardy formula leads to the entropy [33, 36, 37]

$$S = \frac{\pi^2\ell}{3} (C_L T_L + C_R T_R) \quad (32)$$

where $T_L = \frac{r_+ - r_-}{2\pi\ell^2}$, $T_R = \frac{r_+ + r_-}{2\pi\ell^2}$ are the left and right temperatures of BTZ black holes respectively. Thus, we see that when the condition in Eq. (24) holds, the entropy (32) can still be regarded as independent on the AdS radius, even if the cosmological constant is promoted to a variable. This is also consistent with the requirement of differential expression for thermodynamic first law, in which the pressure P and the entropy S should be considered as independent variables to each other.

Finally, there exists the continuous phase transition for BTZ black holes in general 3D gravity models. This is different from that in the normal Einstein gravity, in which BTZ black holes exhibits no interesting phase behaviour [12], since its heat capacity is always positive as verified by the method of thermodynamic curvature [38]. We still take GMMG models for example. Considering the constraints (24), the heat capacity is calculated as

$$C = \frac{\partial M}{\partial T}|_{P,J} = \frac{4\pi(a^2 - b^2)r_+(r_+^2 - r_-^2)\ell}{br_-(3r_+^2 + r_-^2)\ell + ar_+(r_+^2 + 3r_-^2)},$$

where $8G = 1$ is taken for the presentation in Fig.1. It is noted that for our case, only if the parameters a and b have different signs, the thermodynamic process gives the interesting phase transition.

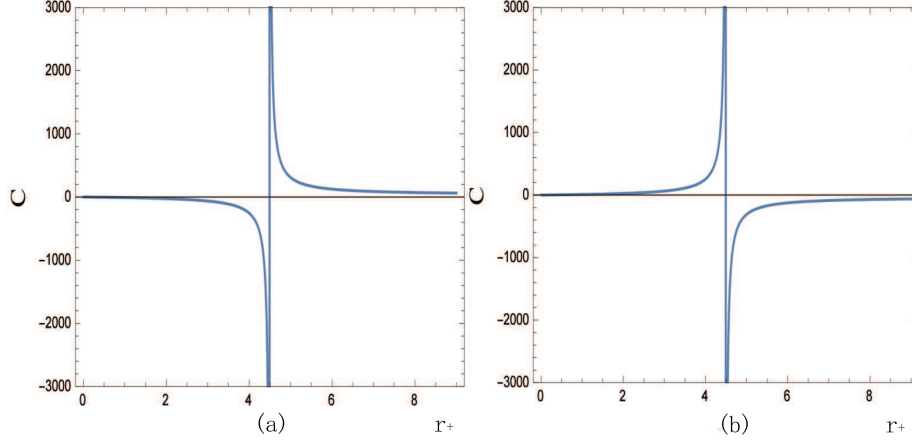


FIG. 1: The heat capacity C as functions r_+ for BTZ black holes in GMMG model, with $\ell = 1$, $r_- = 0.1$, while $a = -2$, $b = 3$ for (a) and $a = 2$, $b = -3$ for (b).

V. CONCLUSION

In this paper, we have investigated homogeneous Smarr relation and the interpretation of thermodynamic enthalpy for the mass of BTZ black holes under the background of 3D gravity theory. For normal 3D BTZ black holes, it is direct to reduce the usual relations of Eqs. (1) and (2) to 3D to obtain the corresponding Smarr formula. For exotic BTZ black holes, it is found that the thermodynamic volume is not geometric and dependent on the angular momentum and the pressure, different from the situation of normal BTZ black holes. However, when we extend these into BTZ black holes in the background of general 3D gravity models, it is impossible to find the proper conjugate thermodynamic volume when considering the variable cosmological constant as the pressure, unless some extra conditions are added. We have studied these conditions for GMMG model, and found that the conditions in Eq. (24) not only ensure both the first law and Smarr formula to hold simultaneously, but also accord with all previous discussions for GMMG model where AdS radius is not regarded as a variable. In particular, the constraint conditions are consistent with the phenomena of continuous phase transition occurred for BTZ black holes in general 3D gravity models.

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